

General production equilibrium with logarithmic functions

Suppose an economy has \bar{K} units of capital and \bar{L} units of labor. There is one firm producing good x and another firm producing good y . Their production functions are

$$x = \ln(L_x) + \ln(K_x)$$

$$y = \ln(L_y) + \ln(K_y)$$

1. Find the contract curve, viewed from sector X and from sector Y
2. Use the previous expressions, together with the production functions and the resource constraints, to find the production possibility frontier

Solution

1. To find the contract curve, we characterize the efficient allocations of capital and labor between the two sectors

The resource constraints are

$$K_x + K_y = \bar{K}$$

$$L_x + L_y = \bar{L}$$

At an interior efficient allocation, the marginal rates of technical substitution must be equal across sectors

For sector X ,

$$x = \ln(L_x) + \ln(K_x)$$

$$MPL_x = \frac{1}{L_x} \quad MPK_x = \frac{1}{K_x}$$

Hence,

$$MRTS_X = \frac{MPL_x}{MPK_x} = \frac{K_x}{L_x}$$

For sector Y ,

$$y = \ln(L_y) + \ln(K_y)$$

$$MPL_y = \frac{1}{L_y} \quad MPK_y = \frac{1}{K_y}$$

Hence,

$$MRTS_Y = \frac{MPL_y}{MPK_y} = \frac{K_y}{L_y}$$

Efficiency requires

$$MRTS_X = MRTS_Y$$

so

$$\frac{K_x}{L_x} = \frac{K_y}{L_y}$$

Using the resource constraints,

$$K_y = \bar{K} - K_x \quad L_y = \bar{L} - L_x$$

we obtain

$$\frac{K_x}{L_x} = \frac{\bar{K} - K_x}{\bar{L} - L_x}$$

Cross-multiplying,

$$K_x(\bar{L} - L_x) = L_x(\bar{K} - K_x)$$

$$K_x\bar{L} - K_xL_x = L_x\bar{K} - L_xK_x$$

$$K_x\bar{L} = L_x\bar{K}$$

Therefore, viewed from sector X , the contract curve is

$$L_x = \frac{\bar{L}}{\bar{K}}K_x$$

or equivalently

$$K_x = \frac{\bar{K}}{\bar{L}}L_x$$

Now write the same condition from sector Y . Since

$$K_y = \bar{K} - K_x \quad L_y = \bar{L} - L_x$$

the same proportionality must hold for sector Y

$$\frac{K_y}{L_y} = \frac{\bar{K}}{\bar{L}}$$

Thus, viewed from sector Y , the contract curve is

$$L_y = \frac{\bar{L}}{\bar{K}}K_y$$

or equivalently

$$K_y = \frac{\bar{K}}{\bar{L}}L_y$$

Hence, the contract curve is the set of allocations in which both sectors use capital and labor in the same proportion as the aggregate endowment ratio

$$\frac{K_x}{L_x} = \frac{K_y}{L_y} = \frac{\bar{K}}{\bar{L}}$$

2. From part (1), along the contract curve the two sectors must use capital and labor in the same proportion as the aggregate endowment ratio

$$\frac{K_x}{L_x} = \frac{K_y}{L_y} = \frac{\bar{K}}{\bar{L}}$$

This implies that there exists some share $\lambda \in (0, 1)$ such that

$$K_x = \lambda\bar{K} \quad L_x = \lambda\bar{L}$$

and therefore

$$K_y = (1 - \lambda)\bar{K} \quad L_y = (1 - \lambda)\bar{L}$$

Now substitute these expressions into the production functions

For sector X ,

$$x = \ln(L_x) + \ln(K_x)$$

$$x = \ln(\lambda\bar{L}) + \ln(\lambda\bar{K})$$

$$x = \ln(\bar{L}\bar{K}) + 2\ln(\lambda)$$

For sector Y ,

$$y = \ln(L_y) + \ln(K_y)$$

$$y = \ln((1 - \lambda)\bar{L}) + \ln((1 - \lambda)\bar{K})$$

$$y = \ln(\bar{L}\bar{K}) + 2\ln(1 - \lambda)$$

Hence, the production possibility frontier can be written parametrically as

$$\begin{cases} x(\lambda) = \ln(\bar{L}\bar{K}) + 2\ln(\lambda) \\ y(\lambda) = \ln(\bar{L}\bar{K}) + 2\ln(1 - \lambda) \end{cases} \quad 0 < \lambda < 1$$

We now eliminate λ in order to obtain y as a function of x

From

$$x = \ln(\bar{L}\bar{K}) + 2\ln(\lambda)$$

we get

$$x - \ln(\bar{L}\bar{K}) = 2\ln(\lambda)$$

$$\ln(\lambda) = \frac{x - \ln(\bar{L}\bar{K})}{2}$$

$$\lambda = \exp\left(\frac{x - \ln(\bar{L}\bar{K})}{2}\right)$$

Substituting into the expression for y ,

$$y = \ln(\bar{L}\bar{K}) + 2\ln\left(1 - \exp\left(\frac{x - \ln(\bar{L}\bar{K})}{2}\right)\right)$$

Therefore, the production possibility frontier is

$$y(x) = \ln(\bar{L}\bar{K}) + 2\ln\left(1 - \exp\left(\frac{x - \ln(\bar{L}\bar{K})}{2}\right)\right)$$

with domain

$$x < \ln(\bar{L}\bar{K})$$

Equivalently, by symmetry, one may also write

$$x(y) = \ln(\bar{L}\bar{K}) + 2 \ln \left(1 - \exp \left(\frac{y - \ln(\bar{L}\bar{K})}{2} \right) \right)$$

Notice that, because the production functions involve logarithms, inputs must be strictly positive in both sectors. Hence the frontier has no finite endpoints. Instead,

$$\lambda \rightarrow 0 \quad \Rightarrow \quad x \rightarrow -\infty, \quad y \rightarrow \ln(\bar{L}\bar{K})$$

and

$$\lambda \rightarrow 1 \quad \Rightarrow \quad x \rightarrow \ln(\bar{L}\bar{K}), \quad y \rightarrow -\infty$$

Thus, the PPF is a decreasing curve asymptotic to the lines

$$x = \ln(\bar{L}\bar{K}) \quad \text{and} \quad y = \ln(\bar{L}\bar{K})$$

Hence, the production possibility frontier is the set of output pairs generated by the parametric system above, or equivalently by the explicit function $y(x)$